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Bounds for indecomposable torsion-free modules

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ABSTRACT

Let M be a finitely generated torsion-free module over a one-dimensional reduced Noetherian ring R with finitely generated normalization. The *rank* of M is the tuple of vector-space dimensions of M_P over each field R_P (R localized at P), where P ranges over the minimal prime ideals of R. We assume that there exists a bound N_R on the ranks of all indecomposable finitely generated torsion-free R-modules. For such rings, what bounds and ranks occur? Partial answers to this question have been given by a plethora of authors over the past forty years. In this article we provide a final answer by giving a concise list of the ranks of indecomposable modules for R a local ring with no condition on the characteristic. We conclude that if the rank of an indecomposable module M is $(r, r, \ldots, r)$, then $r \in \{1, 2, 3, 4, 6\}$, even when R is not local.

1. Introduction

In this article, rings are commutative, Noetherian, one-dimensional, reduced and have finitely generated normalizations. Modules are finitely generated and torsion-free. The *rank* of a module M over a Noetherian reduced ring R is defined to be the ordered tuple of vector-space dimensions $\dim_{R_P} M_P$, where P ranges over the minimal primes of R. If the vector-space dimension of M_P is the same for every minimal prime P of R, we say M has *constant rank*.

What ranks occur for indecomposable modules? Over the past forty years, an extensive series of authors, including Dade, Drozd and Roĭter, Green and Reiner, Jacobinski, and Jones, have studied indecomposable modules over certain local rings, cf. [1–5]. The rings studied classically in the 60s and 70s were one-dimensional domains that were finitely generated over the integers and were contained in algebraic number fields.

Here we consider more general rings of *bounded representation type*, that is, there is a bound on the ranks of the indecomposable finitely generated torsion-free modules. Rings with bounded representation type are also studied by Buchweitz, Greuel and Schreyer in [6] and by Greuel and Knörrer in [7]. Even for such rings, however, the bounds can be arbitrarily large. In 1988, Haefner and Levy showed that, for every positive integer n, there exists a ring of bounded representation type for which an indecomposable module can be constructed having some rank entry n [8].

The first author shows in a previous article that if R is local in addition to the conditions above then the bound on the ranks is 3, provided R satisfies the following condition.

Condition 1.1. R is equicharacteristic with perfect residue field and characteristic not 2, 3 or 5.

He also gives a list of all ranks that occur and examples of indecomposables for each [9].

For R not necessarily local, Chouinard and S. Wiegand in 1991 found that the rank of a constant-rank indecomposable R-module is bounded by 39, cf. [10]. In 1994, R. Wiegand and S. Wiegand improved this bound to 12, cf. [11]. The second

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author, in an article with Arnavut and S. Wiegand, has shown that if R is not necessarily local, but satisfies Condition 1.1, then the bound on the ranks is 6 for constant-rank indecomposable modules, and that, given an integer $n \geq 8$, there are no indecomposable R-modules with all ranks in the interval $[n, 2n-8]$, cf. [12].

In this article, we obtain the theorems of [9,12] without Condition 1.1. Our goal is to prove the following theorem:

Main Theorem 1.2. *Let R be a Noetherian one-dimensional reduced local ring with finitely generated normalization, and assume that there is a bound on the ranks of indecomposable finitely generated torsion-free R-modules. Then R has at most three minimal primes. Moreover:*

1. *If R is a domain, then every indecomposable module has rank 1, 2 or 3.*
2. *If R has exactly two minimal primes, then every indecomposable module has rank*

$$(0, 1), (1, 0), (1, 1), (1, 2), (2, 1) \text{ or } (2, 2).$$

3. *If R has exactly three minimal primes then, with a suitable ordering of the minimal primes, every indecomposable module has rank*

$$(0, 0, 1), (0, 1, 0), (1, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 1, 1) \text{ or } (2, 1, 1).$$

We note that this result does not contradict the results of Haefner and Levy [8] since Main Theorem 1.2 requires that R is a local ring. Main Theorem 1.2 is shown for many cases in the work of Green and Reiner [3, Pages 76–77, 81–82], R. Wiegand and S. Wiegand [11, Theorem 3.9], Çimen [13, Theorem 2.2], Baeth [9, Theorem 4.2], and Arnavut, Luckas and S. Wiegand [12, Theorem 2.3]. In Section 3, we handle the case where an associated residue field extension is separable of degree 2. R. Wiegand and S. Wiegand produced a slightly larger list of possible ranks for the degree 2 separable case, but did not show that all of the ranks on their list occur [11, Theorem 3.10]. They observed that a more careful study of the modules in [3] would be necessary to say exactly what ranks occur. That is our main task in the current work.

In Section 2 we give our setting for this paper as well as relevant background and known results. In Section 3 we show that every module from [11] having rank $(0, 2)$, $(2, 0)$ or $(2, 4)$ decomposes non-trivially. In Section 4 we improve the purely inseparable degree 3 case of the main theorem. In Section 5, we prove our main theorem and give some results for the global case, when R is not necessarily local.

2. Definitions and background

Setting 2.1. Let R be a *ring-order*, that is, a commutative Noetherian one-dimensional reduced ring such that its normalization $\widetilde{R}$ (the integral closure of R in the classic quotient ring of R) is finitely generated as an R-module. We note that every local one-dimensional reduced ring is necessarily Cohen–Macaulay. For the first part of this article we consider local ring-orders R, with maximal ideal $\mathfrak{m}$ and residue field $R/\mathfrak{m} = k$, that have, up to isomorphism, only finitely many indecomposable torsion-free R-modules. Such rings are said to have *finite Cohen–Macaulay type* (abbreviated FCMT), since in our setting, the torsion-free R-modules are exactly the maximal Cohen–Macaulay (abbreviated MCM) R-modules [14, Proposition 0.2]. The local ring-orders of FCMT are characterized as those rings satisfying the two conditions of Drozd and Roĭter.

(dr 1) R has multiplicity at most 3.
(dr 2) $(\mathfrak{m}\widetilde{R} + R)/R$ is cyclic as an R-module.

We say R satisfies (dr) if it satisfies both (dr 1) and (dr 2) above. That (dr) is necessary for finite Cohen–Macaulay type was shown by Wiegand [15, Theorems 2.1 and 3.1]. That (dr) implies finite Cohen–Macaulay type was shown in various cases by Çimen [13, Theorem 1.2], Drozd and Roĭter [2], Green and Reiner [3], Greuel and Knörrer [7], Jacobinski [4], Kiyek and Steinke [16], and Wiegand [17, Theorem 2]. From these results, we have the following theorem:

Theorem 2.2 (*[14, Theorem 0.5]*). *Let R be a one-dimensional Cohen–Macaulay ring.*

1. *If R is local, then there exists a bound on the ranks of indecomposable MCM R-modules if and only if R has FCMT if and only if R is reduced and satisfies (dr).*
2. *If R (not necessarily local) is reduced and the integral closure is finitely generated as an R-module, then R has FCMT if and only if $R_\mathfrak{m}$ has FCMT for every maximal ideal $\mathfrak{m}$ of R.*

Thus the local rings we consider are precisely the Cohen–Macaulay local rings with FCMT. Our goal is to list all possible ranks of indecomposable maximal Cohen–Macaulay modules over these rings. If R has s minimal primes, then every s-tuple of zeros and ones occurs as the rank of an indecomposable maximal Cohen–Macaulay module. For example, if $\{P_1, P_2, P_3\}$ are the minimal primes of R, then the rank of R/P_1 is $(1, 0, 0)$ and the rank of $R/(P_2 \cap P_3)$ is $(0, 1, 1)$. We call such ranks *trivial* sequences. We aim to determine which non-trivial sequences occur as ranks of indecomposable maximal Cohen–Macaulay modules. We may restrict to the case where μ, the multiplicity of R, is less than or equal to 3 by a result of Dade [1]. If $\mu = 1$, however, then R is a DVR and every module is a direct sum of cyclics. If $\mu = 2$, then a result of Bass [18] states that all MCM R-modules are direct sums of ideals, in which case only trivial ranks occur. Thus we consider the case $\mu = 3$. Moreover, it follows from [11, 2.8] and [18, 6.2, 7.2] that we need only consider non-Gorenstein R, since each indecomposable MCM R-module with non-trivial rank is an S-module (necessarily of the same rank) for some local ring S satisfying $R \subset S \subseteq \widetilde{R}$.

For R as in Setting 2.1, we recall the representation of R as an *Artinian pair*, a procedure developed in [3,17,19,11].

Definition 2.3. Consider the following pullback diagram where the vertical maps are the natural projections:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \widetilde{R} \arrow[d] \\ A := R/\mathfrak{c} \arrow[r] & \widetilde{R}/\mathfrak{c} =: B, \end{tikzcd}$$

where $\mathfrak{c}$ is the conductor of R in $\widetilde{R}$, that is, $\mathfrak{c} = \{r \in R \mid r\widetilde{R} \subset R\}$. Assume that $R \neq \widetilde{R}$ to avoid trivial cases. As in the diagram, let $A := R/\mathfrak{c}$ and $B := \widetilde{R}/\mathfrak{c}$. Then $(A \to B)$ is a module-finite extension of Artinian rings. We sometimes denote $(A \to B)$ by R_{art}. A module over the Artinian pair $(A \to B)$ is defined to be a pair of modules $(V \to W)$, where W is a finitely generated projective B-module and V is an A-submodule of W with $BV = W$. Given $(A \to B)$-modules $(V \to W)$ and $(V' \to W')$, a morphism from $(V \to W)$ to $(V' \to W')$ is a B-module homomorphism from W to W' carrying V into V'. We say $(A \to B)$ has *finite representation type* provided there are, up to isomorphism, only finitely many indecomposable $(A \to B)$-modules. Given a MCM R-module M, we denote by M_{art} the R_{art}-module $(M/\mathfrak{c}M \to \widetilde{R}M/\mathfrak{c}M)$ where $\widetilde{R}M$ denotes the torsion-free part of $\widetilde{R} \otimes_R M$.

The following theorem allows us to glean information about an indecomposable MCM R-module M by studying the related indecomposable R_{art}-module M_{art}.

Theorem 2.4 (*[15, 1.6–1.9]*). *Let R be a local ring-order with $R \neq \widetilde{R}$. Let M and N be MCM R-modules, and let $(V \to W)$ be an R_{art}-module.*

1. $(M \oplus N)_{\text{art}} \cong M_{\text{art}} \oplus N_{\text{art}}$.
2. *If $M_{\text{art}} \cong N_{\text{art}}$, then $M \cong N$.*
3. *$(V \to W)$ is isomorphic to X_{art} for some MCM R-module X if and only if $W \cong F/\mathfrak{c}F$ for some projective $\widetilde{R}$-module F.*
4. *The Krull–Schmidt Theorem holds for direct sum decompositions of R_{art}-modules.*
5. *R has FCMT if and only if R_{art} has finite representation type.*

We note that R satisfies (dr) if and only if R_{art} satisfies the following conditions from [11]:

(dr 1) $\dim_k(B/\mathfrak{m}B) \leq 3$.

(dr 2) $\dim_k(\mathfrak{m}B + A)/(\mathfrak{m}^2B + A) \leq 1$.

Notation 2.5. We write $B = B\varepsilon_1 \times \cdots \times B\varepsilon_t$, where the ε_i are the primitive idempotents of B. Let J be the radical of B. Since R satisfies (dr), we have $t \leq 3$, and at most one residue field of B is a proper extension of k. We order the idempotents of B so that $(B/J)\varepsilon_j = k$ for $j < t$, and let $K := (B/J)\varepsilon_t$. This ordering is different from the ordering in [11,13], but agrees with the ordering in [3].

Eliminating residue field growth. Let $(A \to B)$ be the Artinian pair from Definition 2.3, with A local with residue field k, and K a simple extension of k. To handle the situation where K is a proper extension of k, we use techniques of [10, Section 2] and [15, 2.1]: Write $K = k[u]$ for some $u \in K - k$, and let $\bar{f} \in k[x]$ be the minimal polynomial for u over k. Lift $\bar{f}$ to a monic polynomial $f \in A[x]$, and let $A' := \frac{A[x]}{Af}$ and $B' := \frac{B[x]}{Bf}$. By [11, 2.3 (6)], for every indecomposable $(A \to B)$-module M, there exists an indecomposable $(A' \to B')$-module N and an integer $n \leq [K : k]$ such that $M^{(n)} \cong N$ as $(A \to B)$-modules. There is a natural map $(A \to B) \to (A' \to B')$, so that every $(A' \to B')$-module can be viewed as an $(A \to B)$-module. The key to determining the possible ranks of indecomposable $(A \to B)$-modules is to determine how indecomposable $(A' \to B')$-modules decompose over $(A \to B)$.

Green and Reiner [3] give a complete list of the $(A' \to B')$-modules. We use the natural map $(A \to B) \to (A' \to B')$ to determine the possible ranks of indecomposable $(A \to B)$-modules. The matrix reductions of Green and Reiner [3] require that B' be a PIR. However, B' is a PIR as long as K/k is not a degree 2 purely inseparable extension [11, 2.4]. The ranks of modules in this case were computed by Çimen via extensive matrix calculations [13].

Matrix representations. Let $(V \to W)$ be an $(A \to B)$-module, where $(A \to B)$ is an Artinian pair. We represent the module $(V \to W)$ by a matrix. Since W is a finitely generated projective B-module; write $W = (B\varepsilon_1)^{(r_1)} \times \cdots \times (B\varepsilon_t)^{(r_t)}$. The tuple $(r_1, \ldots, r_t)$ is called the *rank* of $(V \to W)$. An element w of W can be written as a vector

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_t \end{bmatrix}$$

where each w_i is itself an $(r_i \times 1)$ column vector in $(B\varepsilon_i)^{(r_i)}$. Take $\{v_1, \ldots, v_n\}$ to be a set of generators for V as an A-module and, since V is a submodule of W, write each v_j as for w,

$$v_j = \begin{bmatrix} v_{1,j} \\ \vdots \\ v_{t,j} \end{bmatrix}.$$

These generators form a matrix F with the v_j making up the columns, that is,

$$F := \begin{bmatrix} v_{1,1} & \ldots & v_{1,n} \\ \vdots & \ddots & \vdots \\ v_{t,1} & \ldots & v_{t,n} \end{bmatrix}.$$

Note that the ith row shown for F is a block of r_i rows, since each $v_{i,j}$ is an $(r_i \times 1)$ column vector. This ith block is an $r_i \times n$ matrix of rank r_i, where here we use rank to mean the dimension of the image of the block. Thus our definition of the rank of $(V \to W)$ as $(r_1, \ldots, r_t)$ matches the usual definition of the rank of a matrix. The column space of F is the module V. Although F is not unique, it determines the module $(V \to W)$ uniquely up to an $(A \to B)$-module isomorphism.

Following [13,3] and [11, page 331] we perform elementary row and column operations that change the matrix without changing the module it determines. We operate within the ith row by doing row operations over $B\varepsilon_i$. We also do column operations by letting $\mathrm{GL}(n, A)$ act on F from the right. This process often allows us to write the corresponding matrix as a direct sum of smaller matrices, thus decomposing the module into a direct sum of modules of smaller ranks.

3. Residue field growth of degree 2

We determine the ranks of the indecomposable MCM modules for the following setting:

Setting 3.1. R is a local ring-order satisfying (dr) and K/k is separable of degree 2. R is non-Gorenstein, $\mu = 3$, and B is a PIR.

The following lemma describes $R_{\text{art}} = (A \to B)$ in more detail.

Lemma 3.2. *Let R be as in Setting 3.1. Then R_{art} is isomorphic to $(A \to k \times B_1)$, where $A = R/\mathfrak{c}$. Moreover, $B_1 \cong A \oplus A\beta$, where β is the root of a monic polynomial $f \in A[x]$ of degree 2, and $\mathrm{Aut}_A(B_1) = \langle \sigma \rangle$, where σ acts on B_1 by permuting the roots of f. Furthermore, if $\operatorname{char} k \neq 2$, we can choose β such that $\beta^2 \in A$.*

Proof. From [11, 2.8], R_{art} has the form $(A \to k \times B_1)$, where $A = R/\mathfrak{c}$. The map $A \to k \times B_1$ is injective. We show the composition $A \to B_1$ of this map with the projection $\Pi : k \times B_1 \to B_1$ is also injective.

The kernel of Π, $k \times (0)$, is a non-zero ideal of $k \times B_1$. Thus the contraction of $k \times (0)$ in A is the kernel of the composition map $A \to B_1$. Since $k \times (0)$ is a simple A-module, the contraction is either zero or all of $k \times (0)$. Since $(A \to k \times B_1)$ is the Artinian pair for the ring R, we have $A = R/\mathfrak{c}$, $B = \widetilde{R}/\mathfrak{c}$. But $\mathfrak{c}$, the conductor of $\widetilde{R}$ into R, is the largest ideal of both R and $\widetilde{R}$. Thus A and B_1 have no non-zero ideals in common. Therefore, the kernel of the composition map $A \to B_1$ is zero, and so, this map is injective.

Now $B_1 = A + A\beta + \mathfrak{n}$, where $\mathfrak{n}$ is the maximal ideal of B_1. Since $\mathfrak{n} = \mathfrak{m}B_1$, we have $B_1 = A + A\beta + \mathfrak{m}B_1$. Nakayama's lemma implies $B_1 = A + A\beta$. By Proposition 2.6 of [11], $\ell(A) = \ell(A\beta) = n$ and $\ell(B_1) = 2n$. Therefore $A \cap A\beta = (0)$, and thus $B_1 = A \oplus A\beta$.

Let $K = k(u) = k[x]/(\bar{f})$, where $f \in A[x]$ is a monic polynomial of degree 2 and where $\bar{f}$ is the image in $k[x]$ of f modulo $\mathfrak{m}$, the maximal ideal of A. Since B_1 is a zero-dimensional local ring, it is complete. Since $\bar{f}$ has two distinct roots in K (the residue field of B_1), by Hensel's lemma [20], f has two distinct roots in B_1. Call one of the roots β. Now $A[x]/(f)$ maps onto B_1. Counting lengths, $\ell(B_1) = \ell(A[x]/(f)) = 2n$ and so $B_1 \cong A[x]/(f)$. Define σ to be the map that fixes A and permutes the two distinct roots of f in B_1. Then $\sigma \in \mathrm{Aut}_A(B_1)$.

Pictorially,

$$\begin{array}{ccc} B_1 = A[x]/(f) = A \oplus A\beta & \twoheadrightarrow & B_1/\mathfrak{n} = K = k(u) \\ \uparrow & & \uparrow \\ A & \twoheadrightarrow & A/\mathfrak{m} = k \end{array}$$

If $\operatorname{char} k \neq 2$, then u can be chosen such that $u^2 \in k$, that is, $\bar{f} = x^2 - u^2$ and $f = x^2 - \beta^2$. $\square$

Let M be an indecomposable $(A \to k \times B_1)$-module. Then by [11, Theorem 2.3], there exists an indecomposable $(B_1 \to K \times B_1 \times B_1)$-module N such that $M \cong N$ as an $(A \to k \times B_1)$-module. Note that N is an $(A \to k \times B_1)$-module, via the map $\Phi_2 : (k \times B_1) \to (K \times B_1 \times B_1)$ defined by

$$\Phi_2 : (a, b) \mapsto (a, b, \sigma(b)) \tag{1}$$

where σ is an A-automorphism of B_1. The indecomposable $(B_1 \to K \times B_1 \times B_1)$-modules are listed in [3, page 82] as matrices. The ranks of these indecomposables are

$$(1,0,0),\ (0,0,1),\ (0,1,0),\ (1,1,0),\ (0,1,1),\ (1,0,1),\ (1,1,1) \text{ and } (2,1,1).$$

A matrix representing a module of rank $(1,0,0)$ is missing from [3]. A module of this rank certainly occurs, however; for example, $(B_1 \to K \times 0 \times 0)$.

As stated in [11], these modules give rise to $(A \to k \times B_1)$-modules of the following ranks:

$$(2,0),\ (0,1),\ (0,1),\ (2,1),\ (0,2),\ (2,1),\ (2,2) \text{ and } (4,2).$$

It is known that there are indecomposable modules of ranks $(1,0)$, $(0,1)$, $(1,1)$, $(2,1)$ and $(2,2)$, cf. [11, Proposition 3.7].

The following lemma shows that there are no indecomposable modules of rank $(0,2)$ or $(2,0)$. We are indebted to Roger Wiegand (private communication) for this proof.

Lemma 3.3. *There are no indecomposable modules of rank* $(2,0)$ *or* $(0,2)$ *over a local ring-order* R *of FCMT.*

Proof. (R. Wiegand) Suppose that R has two minimal primes P and Q. Since $\mu(R) \leq 3$, the multiplicity of R/P is at most 2. Thus, indecomposable torsion-free R/P-modules have rank 1. Now suppose that M is an indecomposable torsion-free R-module of rank $(2,0)$. Then $M_Q = 0$, and, since $PR_P = 0$, we see that $(PM)_P = 0$ and $(PM)_Q = 0$. Since PM is torsion-free, it follows that $PM = 0$. Thus, M is an indecomposable R/P-module of rank 2, a contradiction. That is, there is no indecomposable torsion-free R-module of rank $(2,0)$. By symmetry, there is no indecomposable R-module of rank $(0,2)$ either. $\square$

To show that there are no indecomposable $(A \to k \times B_1)$-modules of rank $(4,2)$, we decompose the $(B_1 \to K \times B_1 \times B_1)$-module, as an $(A \to k \times B_1)$-module, of that rank. The matrix from [3] to be considered is

$$\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 1 & 0 \\ \hline 1 & \pi \end{array}\right],$$

which has rank $(2,1,1)$ as a $(B_1 \to K \times B_1 \times B_1)$-module, and rank $(4,2)$ as an $(A \to k \times B_1)$-module. Here, π is a generator of $\mathfrak{m}$ that also generates the maximal ideal of B_1. We use the convention that $\pi = 0$ in K since π is in the maximal ideal of B_1.

We use the notation $(V' \to W')$ for the $(B_1 \to K \times B_1 \times B_1)$-module. We have $W' = K^2 \times B_1 \times B_1$ and V' is the column space of the matrix, that is,

$$V' = \left\{ \left(\begin{bmatrix} \alpha \\ \xi \end{bmatrix}, \alpha, \alpha + \xi\pi \right) \middle| \alpha, \xi \in B_1 \right\}.$$

V' is an A-module under the action

$$s\left(\begin{bmatrix} \alpha \\ \xi \end{bmatrix}, \alpha, \alpha + \xi\pi \right) = \left(\begin{bmatrix} s\alpha \\ s\xi \end{bmatrix}, s\alpha, \sigma(s)\alpha + \sigma(s)\xi\pi \right),$$

for every $s \in A$. Considering $(V' \to W')$ as the $(A \to k \times B_1)$-module $(V \to W)$, we obtain that $W = k^4 \times (B_1)^2$. Now there is a morphism from $(V \to W)$ to $(V' \to W')$ given by

$$\Theta : W \to W', \qquad \left(\begin{bmatrix} g \\ h \\ j \\ \ell \end{bmatrix}, \begin{bmatrix} p \\ q \end{bmatrix} \right) \mapsto \left(\begin{bmatrix} g + h\beta \\ j + \ell\beta \end{bmatrix}, p, \sigma(q) \right).$$

Note that Θ is compatible with Φ_2 from (1).

Since $B_1 = A \oplus A\beta$ for some $\beta \in B_1$ we write $\alpha = \gamma + \delta\beta$ and $\xi = \nu + \omega\beta$ for $\gamma, \delta, \nu, \omega \in A$, and so, we have the A-module

$$V = \Theta^{-1}\left\{ \left(\begin{bmatrix} \alpha \\ \xi \end{bmatrix}, \alpha, \alpha + \xi\pi \right) \right\} = \left\{ \left(\begin{bmatrix} \gamma \\ \delta \\ \nu \\ \omega \end{bmatrix}, \begin{bmatrix} \gamma + \delta\beta \\ \gamma + \delta\sigma(\beta) + \nu\pi + \omega\sigma(\beta)\pi \end{bmatrix} \right) \middle| \gamma, \delta, \nu, \omega \in A \right\}.$$

Now V is the column space of the matrix

$$\left[\begin{array}{cccc} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ \hline 1 & \beta & 0 & 0 \\ 1 & \sigma(\beta) & \pi & \sigma(\beta)\pi \end{array}\right]. \tag{2}$$

We decompose this rank $(4, 2)$ matrix by letting $GL(A)$ act on the left on the top block, by letting $GL(B_1)$ act on the left on the bottom block, and by letting $GL(A)$ act from the right (on both blocks simultaneously). For the matrix computations, it is important to recall that by Lemma 3.2, there is a degree 2 polynomial $f(x) = x^2 - ax + b$, for some $a, b \in A$, with $f(\beta) = 0$. Since $\sigma(\beta)$ is another root of $f(x)$, we have $\beta + \sigma(\beta) = a \in A$ and $\beta\sigma(\beta) = b \in A$. If, at any stage, a column operation messes up the identity matrix, we can fix it with row operations, since column operations are only over A and row operations in the top block are allowed over A.

If the characteristic of k is 2, then $a \neq 0$, because $f'(x) = -a$ and the separability of the field extension gives $f'(\beta) \neq 0$. Adding the fifth row to the sixth row and replacing $\beta + \sigma(\beta)$ with a gives

$$\left[\begin{array}{cccc} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ \hline 1 & \beta & 0 & 0 \\ 0 & a & \pi & \pi\sigma(\beta) \end{array}\right].$$

Since a is in both A and B_1, we replace the $(6, 2)$ entry with 1 by multiplying column 1 by a^{-1}. Then multiply row 6 by β and add it to row 5. Now multiply column 2 by a^{-1} and add it to column 3 to obtain

$$\left[\begin{array}{cccc} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ \hline 1 & 0 & \beta\pi & \beta\pi\sigma(\beta) \\ 0 & 1 & 0 & \pi\sigma(\beta) \end{array}\right].$$

Substituting b for $\beta\sigma(\beta)$ in the $(5, 4)$ entry, and $a + \beta$ for $\sigma(\beta)$ in the $(6, 4)$ entry gives

$$\left[\begin{array}{cccc} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ \hline 1 & 0 & \beta\pi & b\pi \\ 0 & 1 & 0 & a\pi + \beta\pi \end{array}\right].$$

Since $b\pi \in A$, we can use the first column to clear the $(5, 4)$ entry. Then we add $a\pi$ times column 2 to column 4 to obtain

$$\left[\begin{array}{cccc} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ \hline 1 & 0 & \pi\beta & 0 \\ 0 & 1 & 0 & \pi\beta \end{array}\right] \cong \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 1 & \pi\beta \end{array}\right] \oplus \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 1 & \pi\beta \end{array}\right],$$

the sum of two rank $(2, 1)$ matrices.

If the characteristic of k is not 2, then, by Lemma 3.2, we choose $\beta \in B_1$ such that $\beta^2 \in A$, that is, the irreducible polynomial for β over A is $f(x) = x^2 - \beta^2$. Then $\sigma(\beta) = -\beta$, because $-\beta$ is the other root of f. We decompose the same matrix as in (2), but now it starts out looking like:

$$\left[\begin{array}{cccc} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ \hline 1 & \beta & 0 & 0 \\ 1 & -\beta & \pi & -\pi\beta \end{array}\right].$$

The matrix reductions needed to decompose this matrix are much more straightforward row and column operations and no substitutions are required. After just 2 row operations and 5 column operations we end up with a decomposition into two rank (2, 1) matrices

$$\left[\begin{array}{cccc} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ \hline 1 & 0 & 0 & \pi\beta \\ 0 & \beta & \pi & 0 \end{array}\right] \cong \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 1 & \pi\beta \end{array}\right] \oplus \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline \beta & \pi \end{array}\right].$$

Note that this matrix reduction method can also be used to show that all modules of rank (0, 2) and (2, 0) decompose non-trivially. Section 3 is summarized in the following proposition.

Proposition 3.4. *Let R be a local ring-order with two minimal primes such that R satisfies the Drozd and Roĭter conditions. If K/k is separable of degree 2, then the rank of an indecomposable MCM R-module is one of:* $(0, 1)$, $(1, 0)$, $(1, 1)$, $(2, 1)$ *or* $(2, 2)$.

4. Residue field growth of degree 3

Setting 4.1. R is a local ring-order satisfying (dr) and K/k is purely inseparable of degree 3. R is non-Gorenstein, $\mu = 3$, and B is a PIR.

It was shown in [11] that in this setting all indecomposable torsion-free modules have rank 1, 2 or 3. We show here that all indecomposable torsion-free R-modules have either rank 1 or rank 2. In this case, we have $\mathrm{char}(k) = 3$, and by [11, 2.8], the associated Artinian pair $(A \to B)$ is $(k \to K = k[u])$, where $u^3 \in k$. After eliminating residue field growth, as in [11, page 316], we have $A' = K$ and $B' = K[y]$, where $y^3 = 0$.

To determine the indecomposable modules over $(k \to k[u])$, by [11, Theorem 2.3], we consider indecomposable $(K \to K[y])$-modules as modules over $(k \to k[u])$ using the natural map $\Phi_3 : k[u] \to K[y]$ defined by

$$\Phi_3 : u \mapsto y + u. \tag{3}$$

In [3, page 75], the following list of matrices is given, describing, up to isomorphism, all indecomposable $(K \to K[y])$-modules:

$$G_1 = \begin{bmatrix} 1 & 0 & y \\ 0 & 1 & y^2 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 1 & 0 & y & y^2 \\ 0 & 1 & y^2 & 0 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 1 \end{bmatrix}, \qquad H_2 = \begin{bmatrix} 1 & y \end{bmatrix}, \qquad H_3 = \begin{bmatrix} 1 & y^2 \end{bmatrix}, \qquad H_4 = \begin{bmatrix} 1 & y & y^2 \end{bmatrix}.$$

The first two matrices give $(K \to K[y])$-modules of rank 2 and thus $(k \to k[u])$-modules of rank 6. The last four matrices give $(K \to K[y])$-modules of rank 1, and thus $(k \to k[u])$-modules of rank 3.

The matrix G_1, when considered over $(k \to k[u])$, is shown to decompose into a direct sum of three copies of the rank 2 matrix $\begin{bmatrix} 1 & 0 & -u \\ 0 & 1 & u^2 \end{bmatrix}$, cf. [11, page 332]. This matrix corresponds to a rank 2 module. Thus the $(k \to k[u])$-module given by G_1 decomposes into a direct sum of three rank 2 modules, and we have no rank 3 indecomposable module.

Also G_2, when considered over $(k \to k[u])$, is shown to decompose into a direct sum of two copies of the rank 3 matrix $\begin{bmatrix} 1 & 0 & 0 & u^2 & 0 & u^4 \\ 0 & 1 & 0 & u & u^2 & 0 \\ 0 & 0 & 1 & 0 & u & u^2 \end{bmatrix}$, cf. [11, page 333]. This matrix gives a rank 3 $(k \to k[u])$-module, which we decompose by decomposing the matrix. After some column operations over k and row operations over $K = k[u]$, we have

$$\begin{bmatrix} 1 & 0 & 0 & u^2 & 0 & 0 \\ 0 & 1 & 0 & u & 0 & 0 \\ 0 & 0 & 1 & 0 & u & u^2 \end{bmatrix},$$

which decomposes into the direct sum of a rank 2 and a rank 1 matrix:

$$\begin{bmatrix} 1 & 0 & u^2 \\ 0 & 1 & u \end{bmatrix} \oplus \begin{bmatrix} 1 & u & u^2 \end{bmatrix}.$$

Thus the corresponding rank 3 $(k \to k[u])$-module is the sum of a rank 2 and a rank 1 module.

The matrix $H_1 = [1]$ represents the $(K \to K[y])$-module $(K \to K[y])$, which is $(k+k(u+y)+k(u+y)^2) \to (K \oplus Ky \oplus Ky^2)$ when considered as a $(k \to K)$-module. The matrix $\begin{bmatrix} 1 & u & u^2 \\ 0 & 1 & 2u \\ 0 & 0 & 1 \end{bmatrix}$ represents this rank 3 $(k \to K)$-module. We use row 3 to clear the rest of column 3 since we are allowed row operations over $k[u]$. This yields the decomposition $\begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 \end{bmatrix}$, which is a sum of matrices having ranks 2 and 1.

The matrix $H_2 = \begin{bmatrix} 1 & y \end{bmatrix}$ represents the $(K \to K[y])$-module $((K + Ky) \to K[y])$. Replacing K by $k + k(u+y) + k(u+y)^2$, we get the $(k \to K)$-module

$$((k + k(u + y) + k(u + y)^2) + (k + k(u + y) + k(u + y)^2)y) \to (K \oplus Ky \oplus Ky^2).$$

A matrix representing this is $\begin{bmatrix} 1 & u & u^2 & 0 & 0 & 0 \\ 0 & 1 & 2u & 1 & u & u^2 \\ 0 & 0 & 1 & 0 & 1 & 2u \end{bmatrix}$. After suitable matrix reductions, we have a direct sum of a rank 1 matrix and a rank 2 matrix:

$$\begin{bmatrix} 1 & u \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 & u & 3u^2 \\ 0 & 1 & 1 & 2u \end{bmatrix}.$$

Next we consider $H_3 = \begin{bmatrix} 1 & y^2 \end{bmatrix}$ representing the $(K \to K[y])$-module $(K + Ky^2 \to K[y])$. Replacing K by $k + k(u + y) + k(u + y)^2$, we get the $(k \to K)$-module

$$((k + k(u + y) + k(u + y)^2) + (k + k(u + y) + k(u + y)^2)y^2) \to (K \oplus Ky \oplus Ky^2)$$

which is the column space of the matrix $\begin{bmatrix} 1 & u & u^2 & 0 & 0 & 0 \\ 0 & 1 & 2u & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & u & 2u \end{bmatrix}$. To decompose this matrix, we subtract column 4 from column 3 to obtain a direct sum of a rank 2 matrix and a rank 1 matrix:

$$\begin{bmatrix} 1 & u & u^2 \\ 0 & 1 & 2u \end{bmatrix} \oplus \begin{bmatrix} 1 & u & 2u \end{bmatrix}.$$

To handle H_4, we decompose the matrix

$$\begin{bmatrix} 1 & u & u^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2u & 1 & u & u^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2u & 1 & u & u^2 \end{bmatrix}.$$

After some matrix reductions, we have:

$$\begin{bmatrix} 1 & 0 & 0 & u & 0 & 0 & u^2 & 0 & 0 \\ 0 & 1 & 0 & 0 & u & u^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2u & 0 & u & u^2 \end{bmatrix},$$

which decomposes into a direct sum of a rank 1 matrix and a rank 2 matrix:

$$\begin{bmatrix} 1 & u & u^2 \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 & u & u^2 & 0 & 0 \\ 0 & 1 & 0 & 2u & u & u^2 \end{bmatrix}.$$

The computations of this section prove the following proposition.

Proposition 4.2. *Let R be a local ring-order and a domain such that R satisfies the Drozd and Roĭter conditions. If K/k is purely inseparable of degree* 3*, then the rank of an indecomposable MCM R-module is either* 1 *or* 2 *.*

5. Final summary

As a result of [12,9,13,3,11] and the computations in the previous sections, we have the following theorem.

Main Theorem 5.1. *Let R be a local ring-order satisfying the Drozd and Roĭter conditions. Let M be an indecomposable MCM R-module. Then the rank of M is given in one of the following cases:*

1. *If R has one minimal prime, then the rank of M is 1, 2 or 3.*
2. *If R has two minimal primes, then the rank of M is one of:*

   $(0, 1), (1, 0), (1, 1), (1, 2), (2, 1)$ *or* $(2, 2)$.

3. *If R has three minimal primes, then, for some ordering of the minimal primes, the rank of M is one of:*

   $(0, 0, 1), (0, 1, 0), (1, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 1, 1)$ *or* $(2, 1, 1)$.

Moreover, each of these ranks occur for some appropriate ring R.

Proof. Let $d := [K : k]$, let t be the number of minimal primes of $R_{\text{art}} = A \to B$, and let s be the number of minimal primes of R. Of course, $s \leq t$. As in Section 2, it suffices to consider R non-Gorenstein with $\mu = 3$. By [11, Page 319], $d + s \leq 4$, and hence $d \leq 3$. We consider possible values for d, t and s. If $d = 2$, then $t = 2$ by [11, Theorem 2.8(4)]. If instead $d = 3$, then $t = 1$ by [11, Theorem 2.8(1)].

First assume that $s = t$. Then the ranks of indecomposables over R are just the ranks of indecomposable R_{art}-modules. If $d = 1$, then the theorem holds by the work of Green and Reiner [3, Pages 76–77, 81-82]. There, matrices representing the (finitely many) non-isomorphic indecomposable modules are given. Examples of indecomposable modules of each possible rank can also be found in [11, Theorems 3.2, 3.4, 3.5]. The case of $d = 2$ and K/k purely inseparable is proved in [13, Theorem 2.2]. The case $d = 2$ and K/k separable is shown here in Section 3. The case $d = 3$ and K/k separable but not Galois can be found in [11, Theorem 3.9], with a correction noted in [9, Section 6]. The case $d = 3$ and K/k Galois is proved in [11, Theorem 3.9]. The case $d = 3$ and K/k purely inseparable is shown in [11, Theorem 3.9], and improved here in Section 4.

Now we consider the ranks that occur for M when $s < t$. When $d = 1$, only the ranks in the theorem arise by [11, Theorems 3.2 and 3.4]. The only remaining case is $d = 2$, $t = 2$, and $s = 1$. For an indecomposable R-module M, the R_{art}-module M_{art} is a constant-rank module, and can be decomposed into a direct sum of indecomposable modules of ranks $(1, 0)$, $(0, 1)$, $(1, 1)$, $(1, 2)$, $(2, 1)$ and $(2, 2)$. By considering the possible ranks for indecomposable R_{art}-modules, we see that M_{art} has a direct summand of constant rank 1, 2 or 3. Thus M also does. □

Remark 5.2. We note that, for some ordering of the minimal primes, rank $(1, 2)$ occurs as the rank of an indecomposable only when $d = 1$, and rank $(2, 2)$ occurs only when $d = 1$, or when $d = 2$ and K/k is separable. Further, rank 3 occurs only when $d = 1$, or when $d = 3$ and K/k is separable but not Galois. Indeed, ranks $(2, 2)$ and $(1, 2)$ do not occur in the case $d = 2$ and K/k purely inseparable by [13, Theorem 2.2]. In Section 3, we show that rank $(1, 2)$ does not occur when $d = s = 2$ and K/k separable. It follows that rank 3 does not occur when $d = 2$ and $s = 1$. In Section 4, we show that rank 3 does not occur when $d = 3$ and K/k purely inseparable. Rank 3 also does not occur when K/k is a degree 3 Galois extension, cf. [11, Theorem 3.9].

We now have the following corollaries.

Corollary 5.3. *Let R be a local ring-order satisfying the Drozd–Roĭter conditions. Let M be a MCM R-module of constant rank. Then M has a direct summand of constant rank r for some $r \leq 3$.*

Proof. If R is a domain, then this is part 1 of Main Theorem 1.2. If R has two minimal primes, we write M as a direct sum of indecomposable modules of the ranks given in part 2 of Main Theorem 1.2. Then M has a direct summand of constant rank 1, 2 or 3. Similarly, if R has 3 minimal primes, M has a summand of constant rank 1 or 2. □

Corollary 5.4. *Let R be a local ring-order satisfying the Drozd–Roĭter conditions. Let M be a MCM R-module of constant rank r.*

1. *If $r \geq 3$ and r is odd, then M has a direct summand of constant rank 3.*
2. *If $r = 5$, then M has a direct summand of constant rank 2.*
3. *If $r = 7$, then M has a direct summand of constant rank 4.*
4. *If $r \geq 6$, then M has a direct summand of constant rank 6.*

Proof. By Corollary 5.3, M is a sum of indecomposable modules of constant ranks 1, 2 and 3. We consider all the ways to write various integers as sums of the numbers 1, 2 and 3 to get the results. Items 2 and 3 follow directly from Item 1. □

Discussion: We now establish a theorem about the ranks of indecomposable modules when R is a ring-order that is not necessarily local. We study possible ranks of indecomposable R-modules for a non-local ring R by considering the ranks that occur locally. For example, suppose R is a ring-order with minimal prime ideals $P_1, \ldots, P_s$, and M is an R-module of rank $(r_1, \ldots, r_s)$. Then, if $\mathfrak{m}$ is a maximal ideal of R containing the primes P_1, P_2 and P_3, the rank of $M_{\mathfrak{m}}$ over $R_{\mathfrak{m}}$ is the tuple (r_1, r_2, r_3). Since $R_{\mathfrak{m}}$ is local, (r_1, r_2, r_3) can be written as a sum of ranks occurring for indecomposable modules over a local ring-order. We use [11, Theorem 4.8]: If M_P has a direct summand of rank r for each maximal ideal P, then M has a direct summand of rank r.

It was previously known that the list of possible ranks for the local case given in Main Theorem 1.2 held under Condition 1.1, cf. [9, Theorem 4.2]. The proofs in [12] use Condition 1.1 to do many calculations involving the list of possible ranks for a local ring, however, Condition 1.1 was not used in the proofs for any other reason. Now that we know that Main Theorem 1.2 holds without restriction 1.1 (as suspected all along), Theorem 3.1 from [12] holds without the restriction as well since the calculations go through to the non-local case in exactly the same way.

In the language of [12], a ring R with minimal primes $P_1, \ldots, P_s$ has *bounded representation type* if there exists an integer N_R such that, for each integer i with $1 \leq i \leq s$ and for each indecomposable finitely generated torsion-free R-module M, $\dim_{R_{P_i}}(M_{P_i})$ is less than or equal to N_R. $\text{Int}[a, b]$ is defined to be the set of integers in the closed interval from a to b. We define the *spread* of the rank to be the difference between the largest and smallest rank entry, and we define $\text{Ranks}(M)$ to be the set of rank entries of a module M.

Theorem 5.5. *Let R be a ring-order of bounded representation type, not necessarily local, let n be a positive integer, and let M be a finitely generated torsion-free R-module.*

1. *If* $\mathrm{Ranks}(M) = \{r\}$, *for a positive integer* r, *and* M *is indecomposable, then* $r \in \{1, 2, 3, 4, 6\}$.
2. *If* $n \geq 8$ *and* $\mathrm{Ranks}(M) \subseteq \mathrm{Int}[n, 2n-8]$, *then* M *has a direct summand of constant rank 6.*
3. *For each* $n \geq 8$, *there exist a semilocal ring-order of bounded representation type and a finitely generated indecomposable torsion-free module with rank* $(n, n+1, \ldots, 2n-7)$. *In particular, there exist indecomposable modules with rank* $(8, 9)$, *with rank* $(9, 10, 11)$, *etc.*

Corollary 5.6. *Suppose that* R *is a ring-order of bounded representation type, and* M *is a finitely generated torsion-free indecomposable* R*-module such that the spread of the ranks is* i *for some integer* i. *Then* $\mathrm{Ranks}(M) \subseteq \mathrm{Int}[0, 7+2i]$. *Furthermore, for every integer* $n \geq 8$ *we have* $\mathrm{Ranks}(M) \not\subseteq \mathrm{Int}[n, 2n-8]$. *If the smallest rank entry is also an integer* $j > 8$, *then the spread* i *is at least* $j - 7$.

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